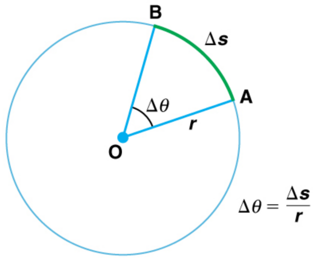
Circular Motion

It is one of the simplest forms of rotational motion. It is a motion of an object about a circle(that is, the axis of rotation passes through the center.) Let's take a point on the circumference of the circle and study it as it rotates. Its distance from the axis doesn't change at all(it is **r**-radius of the circle at all times), however, we see it moving. Thus, to signify this type of rotation, we instead use *angular measures*. Look at the figure below, for instance:



As the object moves from point A to point B on the circumference of the circle, it has deflected its from its initial position by an angle of $\Delta\theta$, while it is still equally as far from its axis of rotation. Thus, to say that our object has rotated is the same as saying that it changed its relative orientation from the axis. Thus, as we discussed displacement as being an objects change in position linearly, we call these change of orientation relative to the axis an **angular displacement**($\Delta\theta$). To explain how fast an object is rotating, our how fast it is changing its orientation, we study the rate of its change in orientation that is:

$$\text{rate of } \Delta\theta = \frac{\Delta\theta}{\Delta t}$$

The above quantity is called angular velocity and it is the time rate of change in angular displacement, and we denote it using the Greek letter Omega(ω).

$$\omega = \frac{\Delta\theta}{\Delta t}$$

The SI-unit of angular displacement is Radian(rad) and the SI-unit of angular velocity is Radian per second(rad/sec). However, we can use multiple other units to describe these quantities. For example, we can use degrees and revolutions to describe angular displacement.

$$1 \text{ rev} = 2\pi rad$$

$$1 \text{ rad} = \frac{180^0}{\pi} \text{ ... thus,}$$

$$1^0 = \frac{\pi \ rad}{180}$$

Similarly as linear velocity changes, we can also have a situation in which the angular velocity is changing. We call this rate of change of angular velocity the angular acceleration(α).

Uniform Circular Motion and Uniformly Accelerated Circular Motion

Uniform Circular Motion is a type of circular motion in which the angular velocity stays constant($\Delta\omega = 0$), that is, the angular acceleration of the object in motion is zero($\alpha = 0$). In that case, we have the following:

$$\omega = \frac{\Delta\theta}{t}$$

We have seen above that $\theta = \frac{S}{r}$, let's try to see the relationship between v and ω.

$$\omega = \frac{\Delta\theta}{t}$$

$$\omega = \frac{\Delta\frac{S}{r}}{t} = \frac{\Delta\frac{S}{t}}{r}$$

$$\omega = \frac{v}{r}$$

Similarly, for acceleration:

$$\alpha = \frac{\Delta\omega}{t}$$

$$\alpha = \frac{\Delta \frac{v}{r}}{t} = \frac{\Delta \frac{v}{t}}{r}$$

$$\alpha = \frac{a}{t}$$

Now, we can discuss Uniformly Accelerated Circular Motion - similarly as in *uniformly accelerated motion*, the angular acceleration of an object in rotational motion stays constant($\Delta\alpha = 0$). In that case, we can use the equations of uniformly accelerated motion by substituting with the angular equivalents of the physical quantities.

$$\omega_f = \omega_i + \alpha t$$

$$\theta = \omega_i t + \frac{\alpha t^2}{2}$$

$$\theta = \omega_f t - \frac{\alpha t^2}{2}$$

$$2\alpha\theta = \omega_f^2 - \omega_i^2$$

$$\theta = \frac{(\omega_f + \omega_i)}{2}$$

When discussing motion, we have seen that we can use calculus to study physical quantities associated with it.

For example,

$$\omega = \frac{\Delta\theta}{\Delta t}$$

When using calculus, we have the following:

$$\omega = \frac{d\theta}{dt}$$

Thus, to find θ in terms of ω, we use integration:

$$d\theta = \omega dt$$

$$\theta = \int \omega dt$$

Doing the same for the acceleration, we have the following:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

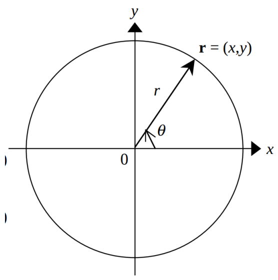
Thus,

$$d\omega = \alpha dt$$

$$\omega = \int \alpha dt$$

Centripetal Acceleration

For an object moving about a circle, the velocity changes instantaneously even when it is rotating in a uniform circular motion. That is, because as an object rotates although its speed may be the same, it changes its direction instantaneously, thus we can safely assume that it is accelerating. Whenever an object is rotating with a constant angular speed, the net force acting on it is called **centripetal force** and the acceleration associated with it is called centripetal acceleration. Understanding the proof for centripetal acceleration here is optional, but **highly recommended** to be read.



Let's consider an object is rotating on the XY plane as shown above. The position of this object at any point on its motion is given by:

$$\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$$

$$\mathbf{r} = rcos\theta\hat{\boldsymbol{i}} + rsin\theta\hat{\boldsymbol{j}}$$

We have seen earlier that:

$$v = \frac{d\mathbf{r}}{dt} = \frac{d(rcos\theta\hat{\boldsymbol{i}} + rsin\theta\hat{\boldsymbol{j}})}{dt} = \frac{d(rcos(\omega t)\hat{\boldsymbol{i}} + rsin(\omega t)\hat{\boldsymbol{j}})}{dt}$$

When we derivate the above equation, we get the following:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -r\omega sin(\omega t)\hat{\boldsymbol{i}} + r\omega cos(\omega t)\hat{\boldsymbol{j}}$$

To find the acceleration, we derivate the above expression one more time with respect to t.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(-r\omega sin(\omega t)\hat{\boldsymbol{i}} + r\omega cos(\omega t)\hat{\boldsymbol{j}})}{dt}$$

We then get the following:

$$\mathbf{a} = -r\omega^2 cos(\omega t)\hat{\boldsymbol{i}} - r\omega^2 sin(\omega t)\hat{\boldsymbol{j}}$$

For a unit vector $\hat{\boldsymbol{r}}$, we know from our our knowledge of vectors that:

$$\hat{\boldsymbol{r}} = \frac{\mathbf{r}}{r} = \frac{rcos\theta\hat{\boldsymbol{i}} + rsin\theta\hat{\boldsymbol{j}}}{r} = cos\theta\hat{\boldsymbol{i}} + sin\theta\hat{\boldsymbol{j}}$$

$$\hat{\boldsymbol{r}} = cos\theta\hat{\boldsymbol{i}} + sin\theta\hat{\boldsymbol{j}} = cos(\omega t)\hat{\boldsymbol{i}} + sin(\omega t)\hat{\boldsymbol{j}}$$

Going back to our equation of acceleration:

$$\mathbf{a} = -r\omega^2 cos(\omega t)\hat{\boldsymbol{i}} + r\omega^2 sin(\omega t)\hat{\boldsymbol{j}}$$

$$\mathbf{a} = -r\omega^2 (cos(\omega t)\hat{\boldsymbol{i}} + sin(\omega t)\hat{\boldsymbol{j}})$$

$$\mathbf{a} = -r\omega^2 (\hat{\boldsymbol{r}})$$

Or if we would like to express this in terms of tangential velocity, we have the following(since $v = \omega r$):

$$\mathbf{a} = -\frac{v^2}{r}(\hat{\boldsymbol{r}})$$

What does the negative sign indicate?

We defined our vector **r** to be outwards from the center and hence the negative of that implies that it is towards the center. Thus, centripetal acceleration is **always** acted towards the center.

For an object moving in a uniform circular motion, we know that it is not accelerating about its axis, that is, $\alpha = 0$. However, it has an acceleration towards the center(the centripetal acceleration - $\boldsymbol{a_c}$). Thus, when we speak of the acceleration of an object while rotating *constantly*, we only talk about the centripetal acceleration.

Let's instead consider an object in a *uniformly accelerated circular motion*, in this case, we have a changing centripetal acceleration at every instant while the angular acceleration is constant($\Delta\alpha = 0$). Thus, if talk about an object in such motion, we are actually talking about the resultant acceleration on the object:

$$\boldsymbol{a_c} = \frac{v^2}{r} = \omega^2 r$$

While we have centripetal acceleration given by the above equation, it is also important to know that we have tangential acceleration as a result of angular acceleration

$$\boldsymbol{a_t} = r\alpha$$

Thus, if we talk about the acceleration of such an object in motion, we talk about the resultant. Since tangential acceleration(tangent) and centripetal acceleration(along the diameter) are always perpendicular, the resultant could be found using the following:

$$\boldsymbol{a} = \sqrt{\boldsymbol{a_c}^2 + \boldsymbol{a_t^2}}$$

$$\boldsymbol{a} = \sqrt{(\omega^2 r)^2 + (r\alpha)^2}$$

When simplified, we get the following:

$$\boldsymbol{a} = r\sqrt{\omega^4 + \alpha^2}$$