Rotational Dynamics

Torque

In this section, we will see interaction of bodies with other objects while rotating and effects of those interactions. Let's start with the simplest case: turning. While studying linear motion, we have seen that the cause of motion is force and it is a result of interaction between objects. The rotational equivalent of force is called **torque** and it depends on three things, one - the amount of force used, two - the distance from the axis of rotation, and three - the inclination of the force on the object(the angle between the force and the axis).

$$\tau = rFsin\theta$$

We can also use vector product to define force. It is the vector product between **r** and **F**(vector product, hence the order is important). It is also important to notice why the equation below is in boldface while the above is not. Recall that we represent vectors using boldface.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Rotation as Seen by Newton's Second Law

In linear motion, Newton's second law states that if an object is accelerating, there is a net force on it or vice versa. It is a bi-implication between force and acceleration.

$$\mathbf{F} \iff \mathbf{a}$$

And we have the following as well:

$$\mathbf{F} = m\mathbf{a}$$

We have seen that Torque is the rotational equivalent of Force and angular acceleration is the rotational equivalent of acceleration(linear). What is then, the rotational equivalent of mass?

$$\boldsymbol{\tau} = ? \times \boldsymbol{\alpha}$$

This unknown physical quantity is called the moment of inertia(I)

Moment of Inertia

In the simplest case possible, we have the following be true:

$$\boldsymbol{\tau} = rF$$

$$\boldsymbol{\tau} = r(m\mathbf{a})$$

$$\boldsymbol{\tau} = r(mr\boldsymbol{\alpha})$$

$$\boldsymbol{\tau} = mr^2\boldsymbol{\alpha}$$

Thus, our unknown in the above section is $\boldsymbol{mr^2}$. This physical quantity is called the rotational inertia of a point mass rotating about an axis **r** meters far from it and with a mass **m**. We use the symbol I to denote moment of inertia. Thus, for a point mass m rotating about an axis at a distance r, we have the moment of inertia be:

$$I = mr^2$$

Thus, our torque equation becomes:

$$\tau = I\alpha$$

Moment of inertia for a point mass and a mass-system with a simple structure is easy to compute. We just add the individual moments to get the total moment of inertia of the system.

$$I = \sum_{1}^{n} m_i r_i^2$$

However, for objects such as a ball and a rod, we can use simple calculus to compute their moments of inertia. Let's start with a rod of mass M and length L that has a uniform linear mass density lambda.

It is imperative to know which axis we are using to rotate the rod to compute its moment of inertia.